Interpolating the Sherrington-Kirkpatrick replica trick[*]

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Abstract

The interpolation techniques have become, in the past decades, a powerful approach to lighten several properties of spin glasses within a simple mathematical framework. Intrinsically, for their construction, these schemes were naturally implemented into the cavity field technique, or its variants as the stochastic stability or the random overlap structures.
However the first and most famous approach to mean field statistical mechanics with quenched disorder is the replica trick.
Among the models where these methods have been used (namely, dealing with frustration and complexity), probably the best known is the Sherrington-Kirkpatrick spin glass:
In this paper we are pleased to apply the interpolation scheme to the replica trick framework and test it directly to the cited paradigmatic model: interestingly this allows to obtain easily the replica-symmetric control and, synergically with the broken replica bounds, a description of the full RSB scenario, both coupled with several minor theorems. Furthermore, by treating the amount of replicas $n \in (0,1]$ as an interpolating parameter (far from its original interpretation) this can be though of as a quenching temperature close to the one introduce in off-equilibrium approaches and, within this viewpoint, the proof of the attended commutativity of the zero replica and the infinite volume limits can be obtained.

1 Introduction

Born as a sideline in the condensed matter division of modern theoretical physics, spin glasses became soon the "harmonic oscillators"[1] of the new paradigm of complexity: hundreds -if not thousands- of papers developed from (and on) this seminal model. Frustration, replica symmetry breaking, rough valleys of free energy, slow relaxational dynamics, aging and rejuvenation (and much more) paved the mathematical and physical strands of a new approach to Nature, where the protagonists are no longer the subjects by themselves but mainly the ways they interact. As a result, complex statistical mechanics is invading areas far beyond condensed matter physics, ranging from biology (e.g. neurology [4, 9, 16] and immunology [7, 30]) to human sciences (e.g. sociology [13, 8] or economics [11, 15]) and much more

[*] *Dedicated to David Sherrington on occasion of his seventieth birthday*

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[1] We learn this beautiful metaphor by Ton Coolen, that we thank.

(see [29] for instance).
Despite a crucial role has been played surely by the underlying graph theory (due to breakthroughs obtained even there, i.e. with the understanding of the small worlds [37] or the scale free networks [3]), we would like to confer to the Sherrington-Kirkpatrick model -SK from now on- (or its concrete variants on graphs, as the Viana-Bray model [36, 25] just to cite one) a crucial role in this new science of complexity.
Among the methods developed for solving its thermodynamics [12, 35], the interpolation techniques, even though not yet so strong to solve the problem in fully autonomy, covered soon a key role to -at least- lighten several properties of this system, working as a synergic alternative to the replica trick [26, 27, 28], which is actually the first and most famous approach to mean field statistical mechanics with quenched disorder: In fact, the interpolation scheme has been "naturally" implemented into the cavity field technique [6, 23, 24], or its variants as the stochastic stability [9, 14, 1] or the random overlap structures [2, 5].
In this paper we want to study this model by extending the interpolating scheme, from the original cavity perspective to the replica trick: To allow this procedure we completely forget the original role played by the "amount" of replicas in the replica trick (tuned by a parameter $n \in (0, 1]$) and think of it directly as a real interpolating parameter. Interestingly this can intuitively though of as a quenching parameter coherently with its counterpart in the glassy dynamics (i.e. FDT violations [17] [18]). At first, once the mathematical strategy has been introduced in complete generality, we use it to obtain a clear picture of the infinite volume and the zero replica limits at the replica symmetric level (by which the whole original SK theory is reproduced), then, within the Parisi full replica symmetry breaking scenario, coupled with the broken replica bounds [21], other robustness properties dealing with the exchange of these two limits are achieved as well.
The paper is therefore structured as follows:
In the next Section, 2, we briefly introduce the model (and the ideas behind the replica trick strategy) while in Section 3 we outline the strategy we want to apply to the model. All the other sections are then left to the implementation of the interpolation into this framework and for presenting the consequent results.

2 The Sherrington-Kirkpatrick mean field spin glass

2.1 The model and its related definitions

The generic configuration of the Sherrington-Kirkpatrick model [26, 27] is determined by the N Ising variables $\sigma_i = \pm 1$, $i = 1, 2, \ldots, N$. The Hamiltonian of the model, in some external magnetic field h, is

$$H_N(\sigma, h; J) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - h \sum_{1 \le i \le N} \sigma_i. \tag{1}$$

The first term in (1) is a long range random two body interaction, while the second represents the interaction of the spins with the magnetic field h. The external quenched disorder is given by the $N(N-1)/2$ independent and identically distributed random variables J_{ij}, defined for each pair of sites. For the sake of simplicity, denoting the average over this disorder by $\mathbb{E}$, we assume each J_{ij} to be a centered unit Gaussian with averages

$$\mathbb{E}(J_{ij}) = 0, \quad \mathbb{E}(J_{ij}^2) = 1.$$

For a given inverse temperature[2] β, we introduce the disorder dependent partition function $Z_N(\beta, h; J)$,

[2]Here and in the following, we set the Boltzmann constant k_{B} equal to one, so that $\beta = 1/(k_{\mathrm{B}}T) = 1/T$.

the quenched average of the free energy per site $f_N(\beta,h)$, the associated averaged normalized log-partition function $\alpha_N(\beta,h)$, and the disorder dependent Boltzmann-Gibbs state ω, according to the definitions

$$\begin{aligned} Z_N(\beta,h;J) &= \sum_\sigma \exp(-\beta H_N(\sigma,h;J)), \qquad (2)\\ -\beta f_N(\beta,h) &= N^{-1}\mathbb{E}\ln Z_N(\beta,h) = \alpha_N(\beta,h), \qquad (3)\\ \omega(A) &= Z_N(\beta,h;J)^{-1}\sum_\sigma A(\sigma)\exp(-\beta H_N(\sigma,h;J)), \qquad (4)\end{aligned}$$

where A is a generic smooth function of σ.

Let us now introduce the important concept of replicas. We consider a generic number n of independent copies of the system, characterized by the spin configurations $\sigma^{(1)},\dots,\sigma^{(n)}$, distributed according to the product state

$$\Omega = \omega^{(1)}\times\omega^{(2)}\times\cdots\times\omega^{(n)},$$

where each $\omega^{(\alpha)}$ acts on the corresponding $\sigma_i^{(\alpha)}$ variables, and all are subject to the *same* sample J of the external disorder.
The overlap between two replicas a,b is defined according to

$$q_{ab}(\sigma^{(a)},\sigma^{(b)}) = \frac{1}{N}\sum_{1\le i\le N}\sigma_i^{(a)}\sigma_i^{(b)}, \qquad (5)$$

and satisfies the obvious bounds $-1\le q_{ab}\le 1$.
For a generic smooth function A of the spin configurations on the n replicas, we define the average $\langle A\rangle$ as

$$\langle A\rangle = \mathbb{E}\Omega A\left(\sigma^{(1)},\sigma^{(2)},\dots,\sigma^{(n)}\right), \qquad (6)$$

where the Boltzmann-Gibbs average Ω acts on the replicated σ variables and $\mathbb{E}$ denotes, as usual, the average with respect to the quenched disorder J.

2.2 The replica trick in a nutshell

The replica trick consists in evaluating the logarithm of the partition function through its power expansion, namely

$$\log Z = \lim_{n\to 0}\frac{Z^n-1}{n} \Rightarrow \langle\log Z\rangle = \lim_{n\to 0}\frac{\langle Z^n\rangle - 1}{n} = \lim_{n\to 0}\frac{1}{n}\log\langle Z^n\rangle, \qquad (7)$$

such that the (intensive) free energy can be written as

$$f_N(\beta,h) = \lim_{n\to 0} f_N(n,\beta,h), \qquad (8)$$

where $f_N(n,\beta,h)$ is defined through

$$-\beta f_N(n,\beta,h) = \alpha_N(n,\beta,h) = \frac{1}{Nn}\log\langle Z^n\rangle. \qquad (9)$$

By assuming the validity of the following commutativity of the n,N limits

$$\lim_{N\to\infty}\lim_{n\to 0}\alpha_N(n,\beta,h) = \lim_{n\to 0}\lim_{N\to\infty}\alpha_N(n,\beta,h) \qquad (10)$$

both Sherrington-Kirkpatrick (at the replica symmetric level [26, 27]) and Parisi (within the full RSB scenario [31, 32, 33]) gave a clear picture of the thermodynamics, which can be streamlined as follows: At the replica symmetric level (i.e. by assuming replica equivalence, namely $q_{ab} = q$ for $a \neq b$, 1 otherwise) we get

$$\alpha_{SK}(\beta) = \min_q \{\alpha(\beta, h, q)\}, \tag{11}$$

where the trial function $\alpha(\beta, h, q)$ is defined as

$$\alpha(\beta, h, q) = \log 2 + \int d\mu(z) \log\cosh\Big(\beta(\sqrt{q}z + h)\Big) + \frac{\beta^2}{4}(1-q)^2. \tag{12}$$

The selfconsistency relation for q reads off as

$$q_{SK} = \int d\mu(z) \tanh^2\Big(\beta(\sqrt{q_{SK}}z + h)\Big)). \tag{13}$$

At the broken replica level we can write

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_N(\beta, J, h) = \alpha(\beta, h) = -\beta f(\beta, h) = \alpha_P(\beta, h), \tag{14}$$

where $\alpha_P(\beta, h)$, the fully broken replica solution, is defined as follows: Let us consider the functional

$$\alpha_P(\beta, h, x) = \log 2 + f(0, y; x, \beta)\mid_{y=h} - \frac{\beta^2}{2}\int_0^1 qx(q)dq, \tag{15}$$

where $f(q, y; x, \beta) \equiv f(q, y)$ is solution of the equation

$$\partial_q f + \frac{1}{2}\partial_y^2 f + \frac{1}{2}x(q)(\partial_y f)^2 = 0, \tag{16}$$

with boundary $f(1, y) = \log\cosh(\beta y)$. Then

$$\alpha_P(\beta, h) = \inf_{x\in\mathcal{X}} \alpha_P(\beta, h, x), \tag{17}$$

where $\mathcal{X}$ is the convex space of the piecewise constant functions as introduced for instance in [21].

3 The interpolating framework for the replica trick

In this Section we present our strategy of investigation; namely we show some Theorems and Propositions whose implications will be exploited in the next Sections. For the sake of clearness we will omit some straightforward demonstrations.

Something close to the replica framework, we can think at the mapping among the one-replica and zero-replica by the introduction of an auxiliary interpolating function -which (non easily) bridges the system among $n = 1$ and $n = 0$- as

$$\varphi_N(n, \beta, h) = \frac{1}{Nn}\log\mathbb{E}(Z_N^n(\beta, J, h)), \tag{18}$$

where, for the sake of clearness $Z_N^n(\beta, J, h) \equiv (Z_N(\beta, J, h))^n$.

It is then worth stressing the next

Theorem 1. *The following relation, among the interpolating function and the free energy, holds*

$$\lim_{n\to 0}\varphi_N(n,\beta,h)=\alpha_N(\beta,h), \tag{19}$$

furthermore

$$\varphi_N(n,\beta,h)\geq\alpha_N(\beta,h) \tag{20}$$

for any n.

Proof. We can expand in Taylor series in $n\in[0,1]$ to get

$$\begin{aligned}\log\mathbb{E}(Z_N^n(\beta,J,h)) &= 0+\mathbb{E}(\log Z_N(\beta,J,h))n+o(n^2)\Rightarrow \\ \lim_{n\to 0^+}\varphi_N(n,\beta,h) &= \lim_{n\to 0}\frac{1}{Nn}(\mathbb{E}(\log Z_N(\beta,J,h))n+o(n^2))=\alpha_N(\beta,h).\end{aligned} \tag{21}$$

The Jensen inequality ensures the second statement of the Theorem. □

Proposition 1. *Through Theorem 1 we immediately obtain*

$$\lim_{N\to\infty}\lim_{n\to 0}\varphi_N(n,\beta,h)=\alpha(\beta,h). \tag{22}$$

We want to deepen now the properties of $\varphi_N(n,\beta,h)$ following the strategy outlined in [20]:

Proposition 2. *Let $i\in\{1,...,N\}$. Introduce positive weights $\forall i\longrightarrow w_i\in\mathbb{R}^+$. Let $\forall i\longrightarrow U_i$ be a family of Gaussian random variables such that $\mathbb{E}(U_i)=0$ and $\mathbb{E}(U_iU_j)=S_{ij}$, where S_{ij} is a positive defined symmetric matrix.*
For the functional $\varphi(n,t)=n^{-1}\log\mathbb{E}(Z_t^n)$, where $Z_t=\sum_i w_i\exp(\sqrt{t}U_i)$, the following relation holds

$$\frac{d}{dt}\varphi(n,t)=\frac{1}{2}\langle S_{ii}\rangle_n+\frac{(n-1)}{2}\langle S_{ij}\rangle_n, \tag{23}$$

where we introduced the following

Definition 1. $\langle A\rangle_n=\mathbb{E}\Big(Z_t^n\mathbb{E}(Z_t^n)^{-1}\Omega(A)\Big)$ *is a deformed state on the* 2*-product Boltzmann one, namely*

$$\Omega(A)=\sum_{i,j}^{N,N}(Z_t^{-1}w_i\exp\sqrt{t}U_i)(Z_t^{-1}\omega_j\exp\sqrt{t}U_j)A,$$

being $A\in\mathcal{A}(Q\times Q)$,

$$\omega(A)=\sum_i^N(Z_t^{-1}w_i\exp\sqrt{t}U_i)A,$$

being $A\in\mathcal{A}(Q)$.

The following generalization, considering two families of random variables, can be easily obtained.

Proposition 3. *Let $i\in Q=\{1,...,N\}$ be a probability space and $\forall i\longrightarrow w_i\in\mathbb{R}^+$ be a probability weight and $\forall i\longrightarrow U_i$ a family of random Gaussian variables such that $\mathbb{E}(U_i)=0$ and $\mathbb{E}(U_iU_j)=S_{ij}$, where S_{ij} is a positive defined symmetric matrix.*
Let $\forall i\longrightarrow\tilde{U}_i$ another family of random Gaussian variables such that $\mathbb{E}(\tilde{U}_i)=0$ and $\mathbb{E}(\tilde{U}_i\tilde{U}_j)=\tilde{S}_{ij}$, where S_{ij} is a positive defined symmetric matrix. Let us further consider the functional $\varphi(n,t)=n^{-1}\log\mathbb{E}(Z_t^n)$ (where $Z_t=\sum_i w_i\exp(\sqrt{t}U_i+\sqrt{1-t}\tilde{U}_i)$): the following relation holds

$$\frac{d}{dt}\varphi(n,t)=\frac{1}{2}\langle S_{ii}-\tilde{S}_{ii}\rangle_n+\frac{(n-1)}{2}\langle S_{ij}-\tilde{S}_{ij}\rangle_n. \tag{24}$$

We can then formulate the following

Theorem 2. *If $\forall (i,j) \in Q \times Q$, $S_{ii} = \tilde{S}_{ii}$ and $S_{ij} \geq \tilde{S}_{ij}$, the following relation holds*

$$\varphi(n,1) \leq \varphi(n,0), \quad \forall n \in (0,1].$$

Proof. Integrating among $0,1$ the functional we get $\varphi(n,1) - \varphi(n,0) = \frac{1}{2}(n-1)\int_0^1 dt \langle S_{ij} - \tilde{S}_{ij} \rangle_n$, whose r.h.s. is ≤ 0 for $n \in (0,1]$.
Obviously the following relation tacitely holds: $\lim_{n\to 0} \langle \cdot \rangle_n = \langle \cdot \rangle$. □

Focusing on the Sherrington-Kirkpatrick model, as earlier introduced, and by using the results of the previous Section, we still think at the n-variation as an interpolation and we can state the following

Theorem 3. *Let us consider the functional $\psi_N(n,\beta,h) = n^{-1}\log \mathbb{E}(Z_N^n(\beta,J,h)) = N\varphi_N(n,\beta,h)$: $\psi_N(n,\beta,h)$ is super-additive in N, $\forall n \in (0,1]$. Furthermore*

$$\lim_{N\to\infty} \varphi_N(n,\beta,h) = \sup_N \varphi_N(n,\beta,h) = \varphi(n,\beta,h), \text{ for any } n.$$

We omit the proof as it is analogous to the one achieved in [22].

Corollary 1. *Remembering that for super-additive (and bounded) functions we can write*

$$\lim_{N\to\infty} \alpha_N(\beta,h) = \sup_N \alpha_N(\beta,h) = \alpha(\beta,h), \tag{25}$$

we get a lower bound for $\varphi(n,\beta,h)$ as $\varphi(n,\beta,h) \geq \alpha(\beta,h)$ and $\sup_N \varphi_N(n,\beta,h) \geq \sup_N \alpha_N(\beta,h)$.

4 Replica symmetric interpolation

For the upper bound we have to tackle the replica symmetric approximation by using a linearization strategy as follows[3]: We introduce and define an interpolating partition function with $t \in [0,1]$ as

$$Z_t = \sum_{\{\sigma\}} \exp(\beta \widetilde{H}(t,\sigma)) \exp\Big(\beta\ h \sum_i^N \sigma_i\Big), \tag{26}$$

where, labeling with $K(\sigma)$ standard $\mathcal{N}(0,1)$ indexed by the configurations σ and characterized by covariance $\mathbb{E}(K(\sigma)K(\sigma')) = q^2_{\sigma\sigma'}$ we defined

$$\widetilde{H}(t,\sigma) = \sqrt{t}\sqrt{\frac{N}{2}}K(\sigma) + \sqrt{1-t}\sqrt{q}\sum_i J_i\sigma_i, \tag{27}$$

where q will play the role of the replica-symmetric overlap, and J_i are random Gaussians i.i.d. $\mathcal{N}[0,1]$ independent even by $K(\sigma)$ and such that

$$\mathbb{E}\Big((\beta\sqrt{q}\sum_i J_i\sigma_i)(\beta\sqrt{q}\sum_j J_j\sigma_j)\Big) = \beta^2 N q q_{\sigma\sigma'}. \tag{28}$$

[3]This procedure is deeply related to the mean field nature of the interactions, which ultimately allows to consider even the low temperature regimes as expressed in terms of high temperature solutions [34]

Lemma 1. *Let us consider the functional* $\varphi(t) = (Nn)^{-1}\log\mathbb{E}(Z_t^n)$*: We have that*

$$\varphi(1) = \frac{1}{Nn}\log\mathbb{E}(Z_1^n) = \varphi_N(n,\beta,h) \tag{29}$$

$$\varphi(0) = \log 2 + \frac{1}{n}\log\int d\mu(z)\cosh^n\Big(\beta(\sqrt{q}z+h)\Big). \tag{30}$$

We are ready to state the next

Theorem 4. $\forall n \in (0,1]$ *we have*

$$\varphi_N(n,\beta,h) \le \log 2 + \frac{1}{n}\log\int d\mu(z)\cosh^n\Big(\beta(\sqrt{q}z+h)\Big) + \frac{\beta^2}{4}(1-2q-(n-1)q^2) \tag{31}$$

uniformly in N*.*

Proof. By applying Proposition 3 we get

$$\frac{d}{dt}\varphi(t) = \frac{\beta^2}{4} - \frac{\beta^2}{2}q + \frac{(n-1)\beta^2}{4}\langle q_{\sigma\sigma'}^2 - 2qq_{\sigma\sigma'}\rangle_n,$$

then, completing with q^2 the square at the r.h.s., and integrating back in $0,1$ we get the thesis. □

In complete analogy with the original SK theory we can define

$$\begin{aligned}\alpha(n,\beta,h,q) &= \log 2 + \frac{1}{n}\log\int d\mu(z)\cosh^n\Big(\beta(\sqrt{q}z+h)\Big) + \frac{\beta^2}{4}(1-2q-(n-1)q^2),\\ \alpha_{RS}(n,\beta,h) &= \min_q(\alpha(n,\beta,h,q)).\end{aligned} \tag{32}$$

Then we get immediately the next

Theorem 5. $\forall n \in (0,1]$*,* $\varphi_N(n,\beta,h) \le \alpha_{SK}(n,\beta,h)$ *uniformly in* N*.*

It is worth noting that the stationarity of q becomes

$$\frac{\partial}{\partial q}\alpha(n,\beta,h,q) = 0 \Rightarrow q_n = \frac{\int d\mu(z)\cosh^n\theta\tanh^2\theta}{\int d\mu(z)\cosh^n\theta} = \langle\tanh^2\theta\rangle_n \tag{33}$$

where we emphasized the n-dependence of q via q_n, we used $\theta = \beta(\sqrt{q_n}z+h)$ for the sake of clearness, $d\mu$ as a standard Gaussian measure and the averages as

$$\langle F\rangle_n = E\Big(\frac{Z^n}{\mathbb{E}(Z^n)}F\Big) = \frac{\int d\mu(z)\cosh^n\theta F}{\int d\mu(z)\cosh^n\theta}.$$

This ensures the validity of the next

Theorem 6. *For all the values of* $n \in (0,1]$ *we have*

$$\alpha_{SK}(n,\beta,h) \ge \alpha_{SK}(\beta,h),\ \lim_{n\to 0}\alpha_{SK}(n,\beta,h) = \alpha_{SK}(\beta,h),$$
$$q_n \ge q_{SK},\quad \lim_{n\to 0} q_n = q_{SK}.$$

Furthermore it is possible to show easily that, under specifical conditions, eq.(33) defines a contraction, implicitly accounting for the high temperature regime[4]. To this task we rewrite the latter as

$$q=\beta^2 q\frac{\int d\theta\exp(-\frac{\theta^2}{2\beta^2 q})\cosh^n\theta\tanh^2\theta}{\int d\theta\exp(-\frac{\theta^2}{2\beta^2 q})\cosh^n(\theta)(\theta-n\beta^2 q\tanh\theta)\theta}, \tag{34}$$

and consider the natural Banach space $\mathcal{B}$ such that $\forall q\in\mathcal{B}\longrightarrow\|q\|\equiv|q|$.
Let us introduce on $\mathcal{B}$ the operator $\mathbf{K}:q\longrightarrow\mathbf{K}(q)$ defined via the original replica symmetric self-consistency relation and use for its norm $\|\mathbf{K}\|\equiv\sup_q(\|\mathbf{K}(q)\|/\|q\|)$. So we can state that

Theorem 7. *$\exists(n,\beta)$: $\boldsymbol{K}$ is a contraction in $\mathcal{B}$ and these are related by $\beta_c(n)=\sqrt{1+n}^{-1}$: coherently with the previous results, criticality is recovered at $\beta_c=1$ when $n\to 0$.*

Proof. By definition

$$\|\mathbf{K}\|=\sup_q\Big\{\frac{\beta^2|q|}{|q|}\frac{|\int d\theta\exp(-\frac{\theta^2}{2\beta^2 q})\cosh^n\theta\tanh^2\theta|}{|\int d\theta\exp(-\frac{\theta^2}{2\beta^2 q})\cosh^n(\theta)(\theta-n\beta^2 q\tanh\theta)\theta|}\Big\}.$$

By using the reversed triangular relation we get $|\tanh\theta|\leq|\theta|\Rightarrow|\theta-n\beta^2 q\tanh\theta|\geq|(|\theta|-n\beta^2 q|\tanh\theta|)|\geq|\theta||1-n\beta^2 q|$ such that

$$\|\mathbf{K}\|\leq\sup_q\Big\{\frac{\beta^2}{|1-n\beta^2 q|}\Big\};\quad q\in[0,1]\Rightarrow\|\mathbf{K}\|\leq\frac{\beta^2}{|1-n\beta^2|}. \tag{35}$$

So if $\beta^2\leq|1-n\beta^2|$, $\mathbf{K}$ is a contraction and $q=0$ is the only solution of the self consistency relation. $\square$

5 Broken replica interpolation

To figure out an easy way to deal with the RSB scenario within an interpolating framework, we now rearrange the scaffold introduced in [20] [21] as follows: Beyond the structures outlines in Propositions 2,3, we introduce $K\in\mathbf{N}$ as an RSB-level counter such that, concretely, $\forall(a,i)$ with $a=1...K$ and $i=1...N$ we manage a family B_i^a of i.i.d. $\mathcal{N}[0,1]$, independent even by the U_i and such that

$$\mathbb{E}(B_i^a B_j^b)=\delta_{ab}\widetilde{S}^a ij. \tag{36}$$

We introduce the averages with respect to the variables $B_i^K,B_i^{K-1}...B_i^1,U_i$ with the notation

$$\mathbb{E}_a(\cdot)=\int d\mu(B_i^a)(\cdot)\ \forall a=1...K,\ \mathbb{E}_0(\cdot)=\int d\mu(U_i)(\cdot),\ \mathbb{E}(\cdot)=\mathbb{E}_0\mathbb{E}_1...\mathbb{E}_K(\cdot),$$

and, $\forall n\in(0,1]$, a family of order parameters $(m_1,...m_K)_n$ with $n<m_a<1\quad\forall a=1,...,K$, and -recursively- the following r.v.

$$Z_K(t)=\sum_i w_i\exp(\sqrt{t}U_i+\sqrt{1-t}\sum_{a=1}^K B_i^a),\ Z_{a-1}^{m_a}=\mathbb{E}_a(Z_a^{m_a}),\quad f_a=\frac{Z_a^{m_a}}{\mathbb{E}_a(Z_a^{m_a})}$$

in perfect analogy with the path outlined in [21]. We are then ready to state the following

[4]High temperature is the β-region where there is only one solution, i.e. $q=0$, of the self-consistency relation: When this condition breaks, phase transition to a broken replica phase appears; we label β_c that particular value of the temperature.

Proposition 4. *Let us consider the functional $\varphi(n,t) = n^{-1}\log E_0(Z_0^n)$. The following relation holds*

$$\frac{d}{dt}\varphi(n,t) = \frac{1}{2}\langle S_{ii} - \widehat{S}_{ii}^{K}\rangle_K^n + \frac{1}{2}\sum_{a=0}^{K}(m_{a+1}-m_a)_n\langle S_{ij} - \widehat{S}_{ij}^{a}\rangle_a^n \tag{37}$$

where $\widehat{S}_{ij}^0 = 0, \qquad \widehat{S}_{ij}^a = \sum_{b=1}^{a}\widetilde{S}_{ij}^b$.

5.1 Lower Bound and Parisi solution

We can apply Proposition 4 to the interpolant $Z_K \equiv Z_t \equiv Z_N(\beta,t,x)$, where

$$Z_N(\beta,t,x) = \sum_{\sigma_1\ldots\sigma_N}\exp\Big(\beta\sqrt{\frac{N}{2}}K(\sigma) + \beta\sqrt{1-t}\sum_{a=1}^{K}\sqrt{q_a-q_{a-1}}J_i^a\sigma_i)\Big)e^{\beta\ h\sum_i\sigma_i}$$

and the J_i^a are defined as the B_i^a (see eq.(36) and above) and x_n mirrors the broken replica steps, namely we introduce a convex space χ_n whose elements are the $x_n(q)$ piecewise functions $x_n : q \to [n,1]$ such that $x_n(q) = m_a(n)$ for $q_{a-1} < q \le q_a \quad \forall a = 1,\ldots,K$, with the prescription $q_0 = 0,\ q_K = 1$.
Note that in this sense we wrote $Z_N(\beta,t,x)$ even though there is no explicit dependence on x at the r.h.s. We then consider the functional

$$\varphi(n,t) = (Nn)^{-1}\log\mathbb{E}_0(Z_0^n) \tag{38}$$

and introduce the following

Lemma 2.

$$\varphi(n,1) = \varphi_N(n,\beta,h), \quad \varphi(n,0) = \log 2 + f(0,h;x_n,\beta),$$

where f satisfies the Parisi equation with x_n as introduced in Section 2.

Consequently the following Theorem holds

Theorem 8. *$\forall n \in (0,1]$ the functional $\varphi(n,t)$ defined in eq.(38) respects the bound*

$$\varphi(n,1) = \varphi_N(n,\beta,h) \le \log 2 + f(0,h;x_n,\beta) - \frac{\beta^2}{4}\Big(1 - \sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2\Big)$$

uniformly in N.

Proof. We can use Proposition 4, keeping in mind the relations

$$\begin{aligned}
&\mathbb{E}\Big(\beta^2\frac{N}{2}K(\sigma)K(\sigma')\Big) = \beta^2\frac{N}{2}q_{12}^2 = S_{ij}, \\
&\mathbb{E}\Big(\beta^2\sqrt{q_a-q_{a-1}}\sqrt{q_b-q_{b-1}}\sum_i J_i^a\sigma_i\sum_j J_j^a\sigma_j\Big), = \beta^2 N(q_a-q_{a-1})q_{12} = \widetilde{S}_{ij}^a.
\end{aligned} \tag{39}$$

to get

$$\frac{d}{dt}\varphi(n,t) = -\frac{\beta^2}{4} - \frac{\beta^2}{4}\sum_{a=0}^{K}(m_{a+1}-m_a)_n\langle q_{12}^2 - 2q_aq_{12}\rangle_a^n.$$

Filling with q^2 the square at the r.h.s. we obtain

$$\frac{d}{dt}\varphi(n,t) = -\frac{\beta^2}{4}(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2) - \frac{\beta^2}{4}\sum_{a=0}^{K}(m_{a+1}-m_a)_n \langle (q_{12}-q_a)^2\rangle_a^n.$$

Lastly, it is enough to remember that

$$(m_{a+1}-m_a)_n \geq 0 \quad \forall a=0,...,K \Rightarrow \varphi(n,1) \leq \varphi(n,0) - \frac{\beta^2}{4}(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2),$$

to get the thesis. □

We can then define

$$\alpha_P(\beta,h,x_n) = \log 2 + n\frac{\beta^2}{4} + f(0,y;x_n,\beta)\mid_{y=h} - \frac{\beta^2}{2}\int_0^1 q x_n(q)dq, \tag{40}$$

and write furthermore that

$$\frac{1}{2}(1-\sum_{a=0}^{K}(m_{a+1}-m_a)_n q_a^2) = \int_0^1 q x_n(q)dq - \frac{n}{2}$$

to state the next

Theorem 9. *The following bounds hold*

$$\begin{aligned} \lim_{N\to\infty} \quad & \varphi_N(n,\beta,h) = \varphi(n,\beta,h) \leq \alpha_P(\beta,h,x_n) \Rightarrow \varphi(n,\beta,h) \leq \inf_{x_n}\alpha_P(\beta,h,x_n), \\ \lim_{n\to 0} \quad & \varphi(n,\beta,h) \leq \lim_{n\to 0}\inf_{x_n}\alpha_P(\beta,h,x_n) = \alpha_P(\beta,h), \end{aligned} \tag{41}$$

and clearly $\lim_{n\to 0}\alpha_P(\beta,h,x_n) = \alpha_P(\beta,h,x)$.

5.2 The temperature of the disorder

In this section we want to try to emphasize the formal analogy between the "real temperature" β and an "effective" temperature n as

$$f(\beta) = \frac{1}{\beta}\mathbb{E}\log\sum_{\sigma} e^{-\beta H(\sigma;J)}, \tag{42}$$

$$f(n) = \frac{1}{n}\log\mathbb{E}e^{n\log Z(J)}. \tag{43}$$

Interestingly for a connection to the dynamical properties of glasses [17] [18], while the Boltzmann temperature β rules the overall energy fluctuations of the system, n seems to tackle the behavior inside the valleys of free energy themselves.
We note that there is a strong correspondence between the approach here performed and the one exploited in [20] whenever n is sent to zero. Furthermore, if we focus on the $n\to 1$ case, we get the annealed regime and the solution for the free energy straightforwardly becomes

$$f_{ann}(\beta,y) = \lim_{N\to\infty}\frac{1}{n}\log\mathbb{E}(Z_N^n)|_{n=1} = \ln 2 + \frac{\beta^2}{4} + \log\cosh(\beta y). \tag{44}$$

As we are interested in thinking at n as an effective temperature selecting valleys of free energies, we stress that by applying the framework we exploited so far, for $n = 1$, χ_n collapses into the space of the constant unitary functions and the solution of eq. (40) coincides with the annealed (44).
We know (see for instance [10]) that mean field spin systems often obey convex representations (trough their order parameters) in temperature. Still bridging, we note that

$$\chi_n \ni x_n : q \to [n,1] \Rightarrow \forall x_n \in \chi_n : \exists x_0 \in \chi_0 : x_n = n x_1 + (1-n) x_0(q).$$

So we see that the space χ_n admits an analogous convex decomposition, with n instead of β: $\chi_n = n\chi_1 \bigoplus (1-n)\chi_0$ [5].

6 The commutativity of $n \to 0$ and $N \to \infty$

In this section we want to deepen our analogy among n and β: the final results would be a proof of the commutativity of the zero replica limit and the infinite volume one, however we need a series of intermediate results.
We star by paying attention at the properties of monotony and convexity of thermodynamical observables with respect to n by considering the usual functional $\varphi(n) = n^{-1} \log \mathbb{E}(Z^n)$ and noticing that, at finite N, Z is regular and we can apply De L'Hopital to get

$$\lim_{n \to 0} \varphi(n) = \mathbb{E}(\log Z) = \langle \log Z \rangle = \varphi(0) \tag{45}$$

which allows to state the next

Theorem 10. *$\varphi(n)$ is a continuous and derivable function in $n \in (0,1]$ and respects the following relations*

$$\varphi(n) = \frac{1}{n} \int_0^n \langle \log Z \rangle_t dt \tag{46}$$

$$\varphi(n) = \langle \log Z \rangle_n + \frac{1}{n} \int_0^n s\, \boldsymbol{Var}_s(\log Z) ds \tag{47}$$

where we used standard definitions as

$$\boldsymbol{Var}_n(X) = \langle X^2 \rangle_n - \langle X \rangle_n^2, \quad \boldsymbol{Var}(X) = \langle X^2 \rangle - \langle X \rangle^2, \tag{48}$$

whose notation represents the, so far, standard averages, namely

$$\langle X \rangle_n = \mathbb{E}(Z^n)^{-1} \mathbb{E}(Z^n X).$$

Proof. With some algebra, directly from the definition of the functional, we get

$$\frac{d\varphi}{dn} = \frac{1}{n} (\langle \log Z \rangle_n - \varphi(n)). \tag{49}$$

Now, coupled together, eq.s (45,49) define a Cauchy problem of the kind

$$\frac{dy}{dx} + p(x) y = q(x)$$

[5] Strictly speaking, in the paper [10] it was shown how to obtain such a decomposition for the free energies. Of course we can expand them in their irreducible overlap correlation functions so to carry on the mapping even at the level of order parameters.

$$y(x_0) = y_0$$

whose general solution is

$$y(x) = \exp\Big(-\int_{x_0}^{x} p(t)dt\Big)\Big\{y_0 + \int_{x_0}^{x} q(t)\exp\Big(\int_{x_0}^{t} p(s)ds\Big)dt\Big\}. \tag{50}$$

We can bypass the singularity in $n = 0$ by considering the slightly modified problem holding for $n \in [\varepsilon, 1)$

$$\frac{d\varphi}{dn} + \frac{1}{n}\varphi(n) = \frac{1}{n}\langle \log Z\rangle_n$$

$$\varphi(\varepsilon) = \varphi_\varepsilon$$

so we can apply the solving kernel (50) getting

$$\varphi(n) = \frac{\varepsilon}{n}\varphi_\varepsilon + \frac{1}{n}\int_{\varepsilon}^{n} \langle \log Z\rangle_t dt$$

and, tacking the $\varepsilon \to 0$ limit obtain a first part of the statement of the Theorem.
Integrating than by parts and remembering that

$$\frac{d}{dn}\langle \log Z\rangle_n = \mathbf{Var}_n(\log Z) \tag{51}$$

we get also eq. (47). □

We note that eq. (47) resembles (in n) the relation

$$f(\beta) = \langle\ H\rangle_\beta - \frac{S(\beta)}{\beta},$$

which pushes further toward our identification of "complexity" with the term coupled with the disorder temperature, namely

$$S(n) = \int_0^n s\mathbf{Var}_s(\log Z)ds \geq 0.$$

As a next step we need then the following

Theorem 11. *$\varphi(n)$ is increasing and convex in n for $n \in [0, 1]$.*

Proof. By using the lastly introduced instruments, we can write

$$\frac{d\varphi}{dn} = \frac{1}{n}\Big(\langle \log Z\rangle_n - \frac{1}{n}\int_0^n \langle \log Z\rangle_t dt\Big)$$

and using the Lagrange Theorem we get

$$\frac{d\varphi}{dn} = \frac{1}{n}\Big(\langle \log Z\rangle_n - \langle \log Z\rangle_\xi\Big),$$

where $\xi \in [0, n]$.
The first part of the Theorem is obtained by noticing that

$$\langle \log Z\rangle_n \geq \langle \log Z\rangle_\xi \Rightarrow$$

while for the second we need to sort out direct calculations:

$$\frac{d\varphi}{dn} = \frac{1}{n}\langle \log Z\rangle_n - \frac{1}{n^2}\log \mathbb{E}(Z^n) \Longrightarrow$$

$$\frac{d^2\varphi}{dn^2} = \frac{2}{n^2}(\varphi(n) - \langle \log Z\rangle_n) + \frac{1}{n}\mathbf{Var}_n(\log Z)_n$$

$$\frac{d^2\varphi}{dn^2} = \frac{2S(n)}{n^3} + \frac{1}{n}\mathbf{Var}_n(\log Z) \geq 0.$$

□

Of course $\varphi_N(n,\beta,h) = \frac{1}{Nn}\log \mathbb{E}(Z_N^n(\beta,J,h))$ respects the hypothesis of regularity at finite N and is then increasing and convex in $n \in [0,1]$. Convexity is preserved in the thermodynamic limit and this implies continuity in the open support $(0,1)$ and uniform convergence in each closed compact belonging to $[0,1]$; the whole suggesting the following

Theorem 12. *The next relation among the infinite volume limit and the zero replica limit is well defined*

$$\lim_{N\to\infty}\lim_{n\to 0^+}\varphi_N(n,\beta,h) = \lim_{n\to 0^+}\lim_{N\to\infty}\varphi_N(n,\beta,h) = \alpha(\beta,h).$$

Proof. As $\varphi_N(n,\beta,h)$ is increasing in n we have that

$$\lim_{N\to\infty}\varphi_N(n,\beta,h)$$

is n-increasing too, then

$$\exists \lim_{n\to 0^+}\lim_{N\to\infty}\varphi_N(n,\beta,h).$$

Of course we have that $\varphi_N(n,\beta,h) \geq \alpha_N(\beta,h) \Rightarrow$

$$\lim_{n\to 0^+}\lim_{N\to\infty}\varphi_N(n,\beta,h) \geq \alpha(\beta,h).$$

To proof the inverse bound we fix the volume (finite) and expand $\varphi_N(n,\beta,h)$ at the first order in Taylor, with the Lagrange remaining term as

$$\varphi_N(n,\beta,h) = \alpha_N(\beta,h) + n\frac{\partial\varphi_N(n)}{\partial n}\Big|_{n=\xi},$$

where $\xi \in [0,n]$.
We can control the derivative with the following argument: For the convexity ofφ, for $n \neq 0$ we have that

$$\lim_{N\to\infty}\varphi_N(n,\beta,h)$$

is continuous, then

$$\frac{\partial\varphi_N(n)}{\partial n}\Big|_{n=\xi}$$

is continuous too and we can choose whatever compact support $\mathcal{C}$ which does not involve the origin $n=0$ where we can apply the Weiestrass Theorem

$$\lim_{N\to\infty}\frac{\partial\varphi_N(n)}{\partial n}\Big|_{n=\xi} \leq \max_{n\in\mathcal{C}}\lim_{N\to\infty}\frac{\partial\varphi_N(n)}{\partial n}\Big|_{n=\xi} = C.$$

Now, remembering that we can ask for convexity, we have that

$$
\begin{aligned}
\lim_{N\to\infty} \frac{\partial \varphi_N(n)}{\partial n}\Big|_{n=\xi} &\leq C \quad \forall \xi \in [0,n] \Rightarrow \\
\lim_{n\to 0^+}\lim_{N\to\infty} \varphi_N(n,\beta,h) &= \lim_{n\to 0^+}\lim_{N\to\infty} \alpha_N(\beta,h) + \lim_{n\to 0^+}\Big(n \lim_{N\to\infty} \frac{\partial \varphi_N(n)}{\partial n}\Big|_{n=\xi}\Big) \\
&\leq \alpha(\beta,h) + C \lim_{n\to 0^+} n \\
&\Rightarrow \lim_{n\to 0^+}\lim_{N\to\infty} \varphi_N(n,\beta,h) \leq \alpha(\beta,h). \qquad (52)
\end{aligned}
$$

□

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